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# A new procedure for calculating periodic response factors based on frequency domain regression method

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**Abstract**

The radiant time series method (RTSM) takes advantages of the fact that design cooling load calculations are based on steady periodic excitations. The main difference between the RTSM and the other cooling load calculation methods is that the periodic response factors of the RTSM are restricted to calculating the conduction heat gain through building elements under periodic outdoor conditions, which simplifies the computational procedure significantly. It is vital to have a reliable method or procedure to accurately calculate the periodic response factors of various types of walls and roofs. In this study, a procedure, based on the frequency-domain regression (FDR) method, is developed to directly and accurately calculate the outside, across and inside periodic response factors of a multilayer wall or roof from its geometric and thermal properties. At first, a polynomial $s$-transfer function is established from the frequency characteristics of the wall or roof using the FDR method. The periodic response factors are then generated from the poles and residues of the polynomial $s$-transfer function. Computational tests show that the FDR method provides an accurate and hopefully better alternative procedure to calculate periodic response factors. Using this procedure, the periodic response factors of various representative wall and roof types are calculated and compared with those calculated by other conventional methods. Some results, particularly of the periodic response factors whose CTF coefficients tabulated in the *ASHRAE Handbook* are inaccurate, are presented and evaluated.


*Keywords:* Frequency domain method; Regression; Periodic response factor; Building heat load calculation

## 1. Introduction

Currently, four methods, i.e. the response factor method, the transfer function method (TFM), the radiant time series method (RTSM) and heat balance method, are available to perform design cooling load calculations. In practice, design cooling load calculations are based on steady periodic outdoor weather condition inputs, but the first two methods have not taken advantage of this fact. The response factor method uses conduction response factors, derived by Mitalas and Stephenson [1] and Hittle [2], to calculate the transient heat conduction through a multilayer wall and roof with boundary conditions that can be represented by a piecewise linear profile. The response factor series is infinite. Therefore, in practice, it must be truncated, resulting in some minor but controllable loss of accuracy. The TFM developed by Stephenson and Mitalas [3,4] uses conduction transfer functions (CTFs) to calculate the transient, one-dimensional heat conduction through the building wall and roof elements. Conduction transfer functions are a closed form representation of a conduction response factor series. The procedures for developing conduction transfer functions from response factors were described by Peavy [5] and Hittle [2]. Obviously, the response factors and conduction transfer functions

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**Nomenclature**

| | | |
|---|---|---|
| $A, B, C, D$ | transmission matrix elements | |
| $a, b, c, d$ | transfer function coefficients | |
| $a_m$ | thermal diffusivity | $m^2 \cdot s^{-1}$ |
| $C_p$ | specific heat | $J \cdot kg^{-1} \cdot K^{-1}$ |
| $G$ | transfer function | |
| $h$ | heat transfer coefficient | $W \cdot m^{-2} \cdot K^{-1}$ |
| $L$ | thickness | M |
| $m$ | term number of denominator | |
| $N$ | number of frequency points | |
| $n$ | layer number of a solid wall | |
| $q, q_\theta, \vartheta$ | heat flow | $W \cdot m^{-2}$ |
| $R$ | thermal resistance | |
| $r$ | term number of numerator | |
| $s$ | Laplace variable or roots | |
| $T$ | temperature | °C or K |
| $t$ | time | s or h |
| $U$ | $U$-factor or thermal transmittance of a wall or roof | $W \cdot m^{-2} \cdot K^{-1}$ |
| $X, Y, Z$ | outside, across and inside response factors | $W \cdot m^{-2} \cdot K^{-1}$ |

*Greek symbols*

| | | |
|---|---|---|
| $\alpha, \beta$ | polynomial $s$-transfer function coefficients | |
| $\Delta\tau$ | time interval | s or h |
| $\delta$ | residue as Eq. (16) | $W \cdot m^{-2} \cdot K^{-1}$ |
| $\mathbf{\Gamma}$ | matrix | |
| $\eta$ | slope of a ramp excitation | $K \cdot h^{-1}$ |
| $\lambda$ | thermal conductivity | $W \cdot m^{-2} \cdot K^{-1}$ |
| $\theta$ | coefficient vector | |
| $\vartheta$ | heat flow for a unit triangular pulse | $W \cdot m^{-2}$ |
| $\mathbf{\Theta}$ | vector | |
| $\rho$ | density | $kg \cdot m^{-3}$ |
| $\omega$ | frequency | $radian \cdot s^{-1}$ |

*Subscripts*

| | |
|---|---|
| $i$ | inside or integer count |
| $j$ | imaginary unit or integer count |
| $k$ | integer count |
| $o$ | outside |
| $P$ | periodic |
| $X, Y, Z$ | outside, across and inside |

impose an unnecessary computational burden on the cooling load calculation procedures.

The Radiant Time Series Method introduced by Spitler et al. [6] takes advantage of the steady periodic nature of design cooling load calculation input parameters. Spitler and Fisher [7] made a comparison between the computational procedures of the TFM and RTSM. In many respects, the RTSM is no different from the TFM described by McQuiston and Spitler [8] and ASHRAE Handbook (1997) [9]. The calculation of solar radiation, transmitted solar heat gain through windows, solar heat gain absorbed by windows, sol–air temperature, and infiltration are exactly the same in both methods. The significant difference between the two methods is the use of periodic response factors in the RTSM. The TFM uses conduction transfer functions to calculate conduction heat gains. The RTSM uses periodic response factors to calculate conduction heat gains. Unlike the transfer function method, which results in a set of equations that must be solved iteratively, the periodic response factor based equations can be solved directly and conveniently on a spreadsheet. Problems related to stability and convergence are avoided and, for most cases, computation time can be reduced. The use of the periodic response factors simplifies the computational procedure.

Accurate and reliable periodic response factors are required for the RTSM to conduct accurate design cooling load calculations. Currently, there are two procedures to generate wall and roof periodic response factors. The first is based on the general conductive response factors of a wall or roof, which are determined directly by Laplace transforms and root-finding procedure. Some measures, as described by Hittle and Bishop [10], might be taken to avoid the root missing for a multilayer wall or roof. The second, which has been developed by Spitler and Fisher [7,11], is based on the CTF coefficients of a wall or roof. The CTF coefficients are essentially developed from response factors, as described by Peavy [5] and Hittle [2]. ASHRAE research project RP-472 provided a set of CTF coefficients corresponding to 41 representative roof types and 42 representative wall types [12]. Spitler and Fisher [7,11] developed their periodic response factors based on the CTF coefficients of all walls and roof types. Of course, if the CTF coefficients of a wall or roof are known and valid, Spitler's procedure is sound and can be used to evaluate the valid periodic response factors. Unfortunately, the CTF coefficients found by conventional methods are not always valid. For instance, the CTF coefficients of some of these wall or roof types provided by ASHRAE are rather inexact. On the other hand, when new building constructions come forth, their CTF coefficients are unknown. Spitler's procedure is not applicable to these constructions. Therefore, there is a need to develop an accurate and hopefully alternative procedure to generate the periodic response factors from the geometric and thermal properties of a wall or roof.

A regression approach is first introduced to derive CTF coefficients of a building envelope from experimental data [13]. A direct frequency-domain regression (DFDR) approach is presented to evaluate CTF coefficients directly from the theoretical frequency characteristics of wall's transient heat conduction [14]. Due to the larger time interval (usually, 3600 seconds) and the nonlinearity of the direct regression approach, the accuracy of the results is a bit lower.

The CTF coefficients of some calculated walls do not meet the features that $b_j$ should be positive and the signs of $d$ should alternate, which is a fundamental requirement. In spite of this, the frequency characteristics of the obtained CTFs have good agreement with the theoretical frequency characteristics of the walls. The CTF coefficients work very well when they are used to calculate heat gain through constructions. Due to its simplicity, the direct regression approach is an alternative way to evaluate wall's CTF coefficients. To improve the accuracy, an improved method—frequency-domain regression (FDR) method is subsequently developed in the paper [15]. Through an intermediate product, polynomial $s$-transfer function evaluated by the FDR method, the response factor and CTF coefficients are calculated.

The FDR method eliminates the limitations of DFDR approach. It results in another limitation, that the three sets of CTF coefficients for a wall do not satisfy the feature that a wall has a unique set of $d$ values. The feature is not practically important since the CTF coefficients are the intermediate results in transient heat flow calculation only and are approximate expressions for the wall transient heat conduction due to the $z$-transform. The heat gain calculated by the CTF coefficients of a wall is the ultimate and important outcome of the wall heat conduction calculation. Only the exact heat gain is required eventually in building thermal analysis and building system simulation. The comparisons and validations through a large amount of calculated examples have fully demonstrated that the hourly heat gain estimated by the CTF coefficients obtained by the FDR method has very good agreement with that estimated by the CTF coefficients obtained by the conventional methods. Another article [16] has reviewed the effectiveness of the FDR method in detail and believed that, despite the limitation of $d$ values, the FDR method is still an accurate and alternative approach to calculate transient heat transfer through multilayer constructions.

In this paper, a new procedure is developed to generate the periodic response factors of multilayer walls and roofs, which is based on the FDR method. In the new procedure, we will observe that there is no relation between CTF coefficients and the calculation of periodic response factors. Thus, there is no need to pay attention to the 'so-called' defect of $d$ value found by the FDR method. This paper is organized into three main sections. First, the procedure for generating the periodic response factors of building constructions is presented. This methodology section introduces the transmission matrix of heat conduction through a multilayer wall or roof and its frequency characteristics, a review of the FDR method to construct a polynomial $s$-transfer function from the frequency characteristics, and the deducing of the calculating formulae for periodic response factors. Secondly, two typical example cases are presented to illustrate and validate the present calculation procedure, and the new procedure is applied to generate the periodic response factors of the representative wall and roof types, whose CTF coefficients are tabulated in *ASHRAE Handbook—Fundamental* (1989, 1993 and 1997) [9,17,18] but are inexact. At last, application guidelines for the new procedure are presented for RTSM users.

## 2. Procedure for generating periodic response factors

In this section, the frequency characteristics of transient heat conduction through a multilayer wall or roof are derived from its transmission matrix. The FDR method is then summarized and used to construct a polynomial $s$-transfer function from the frequency characteristics. Finally, the calculating formulae for periodic response factors are deduced from the polynomial $s$-transfer function.

### 2.1. Transmission matrix of heat conduction through a multilayer construction

Most building walls consist of more than three layers, including the surface air films on both sides. The heat conduction through a building wall can be regarded as a one-dimensional and isothermal process, and each layer of the building walls is homogeneous and isotropic. Considering a solid wall with $n$ layers, the relationship between the temperature and the heat flow on both sides can be expressed as Eq. (1).

$$\begin{bmatrix} T_i(s) \\ q_i(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} T_o(s) \\ q_o(s) \end{bmatrix} \tag{1}$$

where $T(s)$ and $q(s)$ are the Laplace transforms of temperature and heat flow, respectively. Subscripts $i$ and $o$ indicate the inside and outside surfaces of the wall, respectively. The matrix $\left[\begin{smallmatrix} A(s) & B(s) \\ C(s) & D(s) \end{smallmatrix}\right]$ is the total transmission matrix, which is the product of the transmission matrices of all layers, including the surface air films on both sides, as shown in Eq. (2).

$$\begin{aligned} \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} &= \begin{bmatrix} A_i(s) & B_i(s) \\ C_i(s) & D_i(s) \end{bmatrix} \begin{bmatrix} A_1(s) & B_1(s) \\ C_1(s) & D_1(s) \end{bmatrix} \cdots \\ &\quad \times \begin{bmatrix} A_n(s) & B_n(s) \\ C_n(s) & D_n(s) \end{bmatrix} \begin{bmatrix} A_o(s) & B_o(s) \\ C_o(s) & D_o(s) \end{bmatrix} \end{aligned} \tag{2}$$

where $\left[\begin{smallmatrix} A_k(s) & B_k(s) \\ C_k(s) & D_k(s) \end{smallmatrix}\right]$ $(k = 1, 2, \ldots, n)$ is the transmission matrix of the $k$th solid layer. The elements of the transmission matrix of the $k$th layer can be given in the hyperbolic functions of Laplace variable $s$, as shown by Eqs. (3)–(5).

$$A_k = D_k = \cosh\bigl(L_k\sqrt{s/a_{mk}}\bigr) \tag{3}$$

$$B_k = -R_k \sinh\bigl(L_k\sqrt{s/a_{mk}}\bigr)\big/\bigl(L_k\sqrt{s/a_{mk}}\bigr) \tag{4}$$

$$C_k = -L_k\sqrt{s/a_{mk}}\,\sinh\bigl(L_k\sqrt{s/a_{mk}}\bigr)\big/R_k \tag{5}$$

where $L$, $R$ and $a_m (= \lambda/\rho C_P)$ are the thickness, thermal resistance and thermal diffusivity of the corresponding layer, respectively. $\lambda$, $\rho$ and $C_P$ are thermal conductivity, density and specific heat, respectively. When a layer has negligible heat capacity compared to its thermal resistance (e.g., a cavity layer, surface air films), its transmission matrix becomes $\begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}$, where $R$ is the thermal resistance of the cavity layer or surface air film. Thus, the transmission matrices of the inside and outside surface films are $\begin{bmatrix} 1 & -R_i \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -R_o \\ 0 & 1 \end{bmatrix}$.

The total transmission matrix can be rearranged to express the surface heat flows as response and the surface temperature as excitation:

$$\begin{bmatrix} q_o(s) \\ q_i(s) \end{bmatrix} = \begin{bmatrix} -G_X(s) & G_Y(s) \\ -G_Y(s) & G_Z(s) \end{bmatrix} \begin{bmatrix} T_o(s) \\ T_i(s) \end{bmatrix} = \begin{bmatrix} -A(s)/B(s) & 1/B(s) \\ -1/B(s) & D(s)/B(s) \end{bmatrix} \begin{bmatrix} T_o(s) \\ T_i(s) \end{bmatrix} \tag{6}$$

where $G_X(s)$, $G_Y(s)$ and $G_Z(s)$ are the transfer functions of the outside, across and inside heat conduction of the wall, respectively. These matrix elements characterize the dynamic thermal behavior of the wall.

### 2.2. Frequency characteristics

The matrix elements $A(s)/B(s)$, $1/B(s)$ and $D(s)/B(s)$ are the transfer functions of outside, across and inside heat conduction of a multilayer wall, respectively. They are all complicated transcendental hyperbolic functions, especially for the wall of more than two layers. Substituting $j\omega$ ($j = \sqrt{-1}$) for $s$ into Eq. (6), one can obtain the complex functions $G_X(j\omega)$, $G_Y(j\omega)$ and $G_Z(j\omega)$, which are called the frequency characteristics of outside, across and inside heat conduction, respectively, as described by Chen et al. [19]. They are all denoted as $G(j\omega)$. These frequency characteristics are complex functions and generally characterized by their amplitude $|G(j\omega)|$, which is the absolute value of $G(j\omega)$ and phase lag, $\arctan \frac{\mathrm{imag}(G(j\omega))}{\mathrm{real}(G(j\omega))}$, where $\mathrm{real}(G(j\omega))$ and $\mathrm{imag}(G(j\omega))$ are the real and imaginary components of $G(j\omega)$, respectively.

In practice, it is easy to obtain exactly the three frequency characteristics in the frequency domain without finding the embodied expressions of the three complex functions. The calculation approach is as follows. At first, the matrix elements for each layer of the multilayer wall are calculated at $N$ frequency points ($s_k = j\omega_k$, $k = 1, 2, \ldots, N$) by equations (3)–(5). Secondly, the total transmission matrix at each frequency point is obtained by applying matrix multiplication as in Eq. (2). Finally, the three frequency characteristics with $N$ frequency points are established using Eq. (6).

### 2.3. Constructing polynomial $s$-transfer functions

It is much easier and more accurate to obtain the frequency characteristics of a multilayer wall compared with numerically searching for the roots of its characteristic equation. Therefore, the frequency domain regression (FDR) method, as described by Wang and Chen [15,20], is introduced, to develop the periodic response factors of multilayer walls and roofs. Using the FDR method, a few simple $s$-transfer functions are constructed from the frequency characteristics of the calculated wall or roof. This simple $s$-transfer function is the ratio of two polynomials of $s$. For short, it is called the polynomial $s$-transfer function. Using the polynomial $s$-transfer functions, it becomes much easier, simpler and more accurate to generate the periodic response factors and CTF coefficients of a multilayer wall. The constructing procedure for the polynomial $s$-transfer functions is briefed below.

If the properties $\lambda$, $\rho$, $C_p$ and $L$ of each layer in a multilayer wall and the thermal resistance $R_o$ and $R_i$ of its outside and inside surface air films are known, its three frequency characteristics with $N$ frequency points can be easily calculated within the frequency range $[10^{-n_1}, 10^{-n_2}]$, which we need to concern. In general, $n_1 = 7$–$10$, $n_2 = 2$–$4$ and $N = 10(n_1 - n_2) + 1$. The $N$ frequency points are generated with equal logarithmic paces within the frequency range, i.e., $\omega_k = 10^{-n_1+(k-1)(n_1-n_2)/(N-1)}$ ($k = 1, 2, \ldots, N$). The frequency characteristics of the wall at the $k$th point can be expressed as Eq. (7):

$$G(j\omega_k) = P_k + jQ_k \tag{7}$$

The polynomial $s$-transfer function, shown in Eq. (8), can be constructed by the FDR method for each frequency characteristic of the wall or roof.

$$\widetilde{G}(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_r s^r}{1 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_m s^m} = \frac{\widetilde{B}(s)}{1 + \tilde{A}(s)} \tag{8}$$

where $\alpha_i$ and $\beta_i$ are real coefficients, and $r$ and $m$ are the orders of the numerator and denominator, respectively. Generally, $r$ and $m$ are in the range from 4 to 6 and chosen according to the types of the wall or roof. For a lightweight wall or roof, they should be selected as 4, and for a heavyweight one, they should be selected as 6. By substituting $j\omega_k$ for $s$ into Eq. (8), the frequency characteristics of $\widetilde{G}(s)$ at the $k$th point can be expressed as Eq. (9).

$$\widetilde{G}(j\omega_k) = \frac{\beta_0 + \beta_1 j\omega_k + \beta_2 (j\omega_k)^2 + \cdots + \beta_r (j\omega_k)^r}{1 + \alpha_1 j\omega_k + \alpha_2 (j\omega_k)^2 + \cdots + \alpha_m (j\omega_k)^m} = \frac{\widetilde{B}(j\omega_k)}{1 + \tilde{A}(j\omega_k)} \tag{9}$$

By minimizing the sum of the square error between the frequency characteristics of the wall and the polynomial $s$-transfer function at all frequency points, the coefficients of the polynomial $s$-transfer function are easily obtained by solving a set of linear equations as Eq. (10).

$$\boldsymbol{\theta} = \boldsymbol{\Gamma}^{-1} \boldsymbol{\Theta} \tag{10}$$

where

$$\boldsymbol{\theta}^{\mathrm{T}} = [\,\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \ldots \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \ldots\,] \tag{11}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} v_0 & 0 & -v_2 & 0 & v_4 & \ldots & w_1 & \chi_2 & -w_3 & -\chi_4 & w_5 & \ldots \\ 0 & v_2 & 0 & -v_4 & 0 & \ldots & -\chi_2 & w_3 & \chi_4 & -w_5 & -\chi_6 & \ldots \\ -v_2 & 0 & v_4 & 0 & -v_6 & \ldots & -w_3 & -\chi_4 & w_5 & \chi_6 & -w_7 & \ldots \\ 0 & -v_4 & 0 & v_6 & 0 & \ldots & \chi_4 & -w_5 & -\chi_6 & w_7 & \chi_8 & \ldots \\ v_4 & 0 & -v_6 & 0 & v_8 & \ldots & w_5 & \chi_6 & -w_7 & -\chi_8 & w_9 & \ldots \\ \vdots & & & & & \ddots & & & & & & \\ w_1 & -\chi_2 & -w_3 & \chi_4 & w_5 & \ldots & u_2 & 0 & -u_4 & 0 & u_6 & \ldots \\ \chi_2 & w_3 & -\chi_4 & -w_5 & \chi_6 & \ldots & 0 & u_4 & 0 & -u_6 & 0 & \ldots \\ -w_3 & \chi_4 & w_5 & -\chi_6 & -w_7 & \ldots & -u_4 & 0 & u_6 & 0 & -u_8 & \ldots \\ -\chi_4 & -w_5 & \chi_6 & w_7 & -\chi_8 & \ldots & 0 & -u_6 & 0 & u_8 & 0 & \ldots \\ w_5 & -\chi_6 & -w_7 & \chi_8 & w_9 & \ldots & u_6 & 0 & -u_8 & 0 & u_{10} & \ldots \\ \vdots & & & & & & & & & & & \end{bmatrix} \tag{12}$$

$$\boldsymbol{\Theta}^{\mathrm{T}} = [\,\chi_0 \quad w_1 \quad -\chi_2 \quad -w_3 \quad \chi_4 \quad \ldots \quad 0 \quad u_2 \quad 0 \quad u_4 \quad 0 \quad \ldots\,] \tag{13}$$

$$\left.\begin{array}{ll} v_i = \displaystyle\sum_{k=1}^{N} \omega_k^i, & \chi_i = \displaystyle\sum_{k=1}^{N} \omega_k^i P_k \\ w_i = \displaystyle\sum_{k=1}^{N} \omega_k^i Q_k, & u_i = \displaystyle\sum_{k=1}^{N} \omega_k^i \left(P_k^2 + Q_k^2\right) \end{array}\right\} \tag{14}$$

It is should be noted that the matrix $\boldsymbol{\Gamma}$ may be close to singular. Directly inverting matrix $\boldsymbol{\Gamma}$ might lead to inaccurate result. To improve computational accuracy, the pseudoinverse of matrix $\boldsymbol{\Gamma}$ based on singular value decomposition should be used to solve Eq. (10). $n_1$ and $n_2$ should be selected correctly for the different kinds of building constructions, such as light, medium and heavy weight walls. The value of $n_1$ should be selected within the range of 7 to 10, and $n_2$ be done within the range of 2 to 4. For the cross heat conduction, the heavier the building constructions, the greater the values of $n_1$ and $n_2$ should be. Sometimes, in order to improve the computational accuracy, the frequency characteristics from the $N_1$th to $N_2$th frequency point within the $N$ frequency points are selected to construct the polynomial $s$-transfer functions, where $N_1 \geqslant 1$ and $N_2 \leqslant N$. During computation, the values of $N_1$ and $N_2$ should be adjusted until the computational accuracy is not improved any more. The coefficients of polynomial $s$-transfer functions change with the difference of parameters $n$, $m$, $n_1$, $n_2$, $N_1$ and $N_2$. However, a little change takes place in the periodic response factors based on the polynomial $s$-transfer functions for the different parameters.

### 2.4. Formulae for periodic response factors

The periodic response factors of across heat conduction are chosen as an example to explain the procedure. After the polynomial $s$-transfer function $\widetilde{G}_Y(s)$ for the across heat conduction has been constructed, it is easy to find the $m$ roots ($s_i$, $i = 1, 2, \ldots, m$) of its denominator $1 + \tilde{A}(s)$ (i.e., the $m$ poles of $\widetilde{G}_Y(s)$). A ramp excitation is defined as an increase at time $t = 0$ with a slope of $\eta = 1\ \mathrm{K{\cdot}h^{-1}}$ in the outside air temperature of a wall, which is at zero temperature everywhere before that time and whose inside air temperature is subsequently maintained at zero. Supposing that a wall is imposed on by such a unit ramp excitation ($\eta = 1$) and that the roots of $1 + \tilde{A}(s)$ are non-repeated, the heat flow $q_Y(\tau)$ ($\mathrm{W{\cdot}m^{-2}{\cdot}K^{-1}}$) at the inside surface of the wall (i.e., the response of the polynomial $s$-transfer function $\widetilde{G}_Y(s)$ to the unit ramp excitation is given by Eq. (8).

$$\begin{aligned} q_Y(t) &= \mathbf{L}^{-1}\left(\frac{\widetilde{G}_Y(s)}{s^2}\right) = \mathbf{L}^{-1}\left(\frac{\widetilde{B}(s)}{s^2(1 + \tilde{A}(s))}\right) \\ &= Ut + \sum_{i=1}^{m} \delta_i \left(1 - e^{s_i t}\right) \end{aligned} \tag{15}$$

where, $U$ is the $U$-factor or thermal transmittance of the wall or roof, $\delta_i$ ($i = 1, 2, \ldots, m$) ($\mathrm{W{\cdot}m^{-2}{\cdot}K^{-1}}$) is the residue of $G_Y(s)/s^2$ corresponding to the $i$th root and can be calculated as Eq. (16).

$$\delta_i = -\widetilde{B}(s_i)/\left[s_i^2 \dot{\tilde{A}}(s_i)\right] \tag{16}$$

Where $\dot{\tilde{A}}(s_i)$ is the derivative of $\tilde{A}(s)$ at the $i$th root. That is $\dot{\tilde{A}}(s_i) = \alpha_1 + 2\alpha_2 s_i + \cdots + m\alpha_m s_i^{m-1}$. It should be noted that Eqs. (15) and (16) are only valid for non-repeated roots.

Since a unit triangular pulse, which is of height $\phi = 1$ K and base $2\Delta\tau$ at time $t = 0$, can be formed by a unit ramp (at time $t = -\Delta\tau$), a $\eta = -2$ ramp (at time $t = 0$) and a unit ramp (at time $t = \Delta\tau$), the heat flow on the inside surface of the wall due to a unit triangular pulse is calculated by Eq. (17).

$$\begin{aligned} \vartheta_Y(t) &= \frac{1}{\Delta\tau}\left(q_Y(t + \Delta\tau) - 2q_Y(t) + q_Y(t - \Delta\tau)\right) \\ &= -\sum_{i=1}^{m} \frac{\delta_i}{\Delta\tau}\left(1 - e^{s_i\Delta\tau}\right)^2 e^{s_i(t-\Delta\tau)} \end{aligned} \tag{17}$$

The response factors themselves, $Y_j$ ($j = 0, 1, 2, 3, \ldots$), are the values of $\vartheta_Y(t)$ at time $t = j\Delta\tau$ ($j = 0, 1, 2, 3, \ldots$). Conventionally, $\Delta\tau = 1$ hour. The value of the first factor, $Y_0$, is derived from a single unit ramp at time $t = 0\Delta\tau$, as shown in Eq. (18).

$$\begin{aligned} Y_0 &= \frac{1}{\Delta\tau} q_Y(0\Delta\tau + \Delta\tau) \\ &= \frac{1}{\Delta\tau}\left(U\Delta\tau + \sum_{i=1}^{m} \delta_i\left(1 - e^{s_i\Delta\tau}\right)\right) \\ &= U + \sum_{i=1}^{m} \frac{\delta_i}{\Delta\tau}\left(1 - e^{s_i\Delta\tau}\right) \end{aligned} \tag{18}$$

The subsequent factors $Y_j$ ($j = 1, 2, 3, \ldots$) are derived from the superposition $\vartheta_Y(t)$ of the three ramps as shown in Eq. (19).

$$Y_j = \vartheta_Y(j\Delta\tau) = -\sum_{i=1}^{m} \frac{\delta_i}{\Delta\tau}\left(1 - e^{s_i\Delta\tau}\right)^2 e^{(j-1)s_i\Delta\tau} \tag{19}$$

The response factors $X_j$ and $Z_j$ ($j = 0, 1, 2, 3, \ldots$) can be calculated using the same formulae as Eqs. (18) and (19) from the polynomial $s$-transfer functions $\widetilde{G}_X(s)$ and $\widetilde{G}_Z(s)$, which are constructed respectively from the transfer functions $G_X(s)$ and $G_Z(s)$ using the FDR method. The heat conduction through a wall can be represented by the general response factors in Eq. (20).

$$q_\theta = -\sum_{j=0}^{n} Z_j T_{i,t-j\Delta\tau} + \sum_{j=0}^{n} Y_j T_{o,t-j\Delta\tau} \tag{20}$$

If the boundary conditions are steady periodic and of a 24-hour period, Eq. (20) can be rewritten as

$$q_\theta = -\sum_{j=0}^{23} Z_{Pj} T_{i,t-j\Delta\tau} + \sum_{j=0}^{23} Y_{Pj} T_{o,t-j\Delta\tau} \tag{21}$$

where $Y_{Pj}$ and $Z_{Pj}$ are called the periodic response factors. They are designated to be either inside-coefficients ($Z$) or across-coefficients ($Y$), depending on the temperature by which they are multiplied. The set of periodic response factors ($Y_{Pj}$ and $Z_{Pj}$) can be represented as Eqs. (22) and (23).

$$Y_{Pj} = Y_j + Y_{j+M} + Y_{j+2M} + \cdots \quad (j = 0, 1, 2, \ldots, M-1) \tag{22}$$

$$Z_{Pj} = Z_j + Z_{j+M} + Z_{j+2M} + \cdots \quad (j = 0, 1, 2, \ldots, M-1) \tag{23}$$

where $M = 24$. According to Eqs. (18) and (19), across periodic response factors ($Y_{Pj}$) can be expressed as Eqs. (24) and (25).

$$Y_{P0} = U + \sum_{i=1}^{m} \frac{\delta_i}{\Delta\tau} \left(1 - e^{s_i\Delta\tau}\right) \frac{1 - e^{(M-1)s_i\Delta\tau}}{1 - e^{Ms_i\Delta\tau}} \tag{24}$$

$$Y_{Pj} = -\sum_{i=1}^{m} \frac{\delta_i}{\Delta\tau} \left(1 - e^{s_i\Delta\tau}\right)^2 \frac{e^{(j-1)s_i\Delta\tau}}{1 - e^{Ms_i\Delta\tau}} \quad (1 \leqslant j \leqslant M-1) \tag{25}$$

Similarly, the outside and inside periodic response factors ($X_{Pj}$ and $Z_{Pj}$, $j = 0, 1, 2, \ldots, M-1$) can be calculated respectively from the poles and residues of the polynomial $s$-transfer function $\widetilde{G}_X(s)$ and $\widetilde{G}_Z(s)$. Their calculating formulae are the same as in Eqs. (24) and (25). It should be noted that the $U$-factor of a wall or roof is equal to the sum of its periodic response factors, i.e., $U = \sum X_P = \sum Y_P = \sum Z_P$ [11]. This important property can be used to check whether the periodic response factors of a wall or roof are accurate or not.

It is easy and accurate to calculate the frequency characteristics of a wall or roof and to construct its polynomial $s$-transfer functions from its frequency characteristics. It is also easy to calculate the poles and residues of the polynomial $s$-transfer functions. Therefore, it is certainly easier and simpler to generate the periodic response factors of a wall or roof by using the FDR method.

In order to generate the periodic response factors, in the following computational tests, the following steps are implemented in a MATLAB program:

- Input the thickness and thermal properties of all layers in a wall or roof;
- Calculate its frequency characteristics by matrix multiplication;
- Construct polynomial $s$-transfer function;
- Calculate the poles and residues of the polynomial $s$-transfer function;
- Generate periodic response factors.

## 3. Comparisons and validations

Various walls and roofs used in references and handbooks are tested to validate the new calculation procedure based on the FDR method. The results for two typical walls and a few problematic walls (used in ASHRAE handbooks) are presented below and compared with the periodic response factors generated from their CTF coefficients.

### *3.1. Results of typical walls*

#### *3.1.1. A common brick wall*

The wall consists of an outside air film ($h_o = 18.3$ W·m$^{-2}$·K$^{-1}$), a layer of common brickwork, a layer of plaster and an inside air film ($h_i = 8.7$ W·m$^{-2}$·K$^{-1}$) as described in Table 1. By directly using Laplace transform methods, the CTF coefficients of the common brick wall are determined as listed in Table 2. In this example, some intermediate results are provided here to assist readers in verifying the present approach. The total transmission matrix of the wall is provided in form of matrix multiplication, shown as Eq. (26). The fifth-order polynomial $s$-transfer function $G_Y(s)$ for cross heat conduction is found by FDR method and given in Eq. (27). In the computation, the first

Table 1
Details of a common brick wall

| Description | Thickness and thermal properties | | | | |
|---|---|---|---|---|---|
| | $L$ [mm] | $\lambda$ [W·m$^{-1}$·K$^{-1}$] | $\rho$ [kg·m$^{-3}$] | $C_P$ [J·kg$^{-1}$·K$^{-1}$] | $R$ [m$^2$·K·W$^{-1}$] |
| Outside air film | | | | | 0.054645 |
| Brickwork | 240 | 0.810 | 1800 | 880 | 0.296296 |
| Plaster | 20 | 0.700 | 1600 | 880 | 0.028571 |
| Inside air film | | | | | 0.114943 |

47 points of $N = 10 \times (8-3)+1$ frequency points within the frequency rang of $10^{-8}$ to $10^{-3}$ radian·s$^{-1}$ are used to construct the polynomial $s$-transfer function. Its hourly heat flow $\vartheta_Y(j)$ for a unit triangular temperature pulse is calculated using Eq. (17) and listed in Table 5. Its across periodic response factors are calculated using the present procedure and the CTF coefficients, and are listed in Table 6 for comparison. It can be found that there is no difference between the $U$-factor of the wall and the sum of the periodic response factors generated by the present procedure, and that the periodic response factors are identical to those calculated using the CTF coefficients

$$\begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{8.7} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cosh(28.3650\sqrt{s}) & -\frac{1.0072\times 10^{-3}\sinh(28.3650\sqrt{s})}{\sqrt{s}} \\ -992.7739\sqrt{s}\sinh(28.3650\sqrt{s}) & \cosh(28.3650\sqrt{s}) \end{bmatrix} \times \begin{bmatrix} \cosh(335.6188\sqrt{s}) & -\frac{8.8283\times 10^{-4}\sinh(335.6188\sqrt{s})}{\sqrt{s}} \\ -1132.7135\sqrt{s}\sinh(335.6188\sqrt{s}) & \cosh(335.6188\sqrt{s}) \end{bmatrix} \times \begin{bmatrix} 1 & -\frac{1}{18.3} \\ 0 & 1 \end{bmatrix} \tag{26}$$

$$\begin{aligned} \widetilde{G}_Y(s) = {} & \big[-1.637148\times 10^{-4} + 1.966158\times 10^{-6}s \\ & - 4.949933\times 10^{-9}s^2 + 7.147446\times 10^{-12}s^3 \\ & - 6.217503\times 10^{-15}s^4 + 2.625507\times 10^{-18}s^5\big] \\ & \times \big[1.0 + 2.031942\times 10^{-3}s + 1.521376\times 10^{-6}s^2 \\ & + 4.701313\times 10^{-10}s^3 + 5.214346\times 10^{-14}s^4 \\ & + 1.298195\times 10^{-18}s^5\big]^{-1} \end{aligned} \tag{27}$$

### 3.1.2. A brick/cavity wall

Davies [21] considered a brick/cavity wall as described in Table 3 (*CIBSE 1986, Guide Book Volume A* [22], P. A3-25, Example 4, omitting the plaster), and provided all its transfer function coefficients (listed in Table 4) using time-domain methods. The outside, across and inside periodic response factors were all determined by the procedure based on the FDR method and the CTF coefficients respectively, and are listed in Table 6 for comparison. It can be found that there is also very good agreement between the periodic response factors generated by both methods. The sums of all periodic response factors ($\sum X_P$, $\sum Y_P$ and $\sum Z_P$) generated by the present procedure are even closer to the $U$-factor of the wall.

## 3.2. Results of problematic walls

Harris and McQuiston [12] developed conduction transfer function CTF coefficients corresponding to 41 representative wall assemblies and 42 representative roof assemblies for the application of TFM. They also developed a grouping procedure that allows design engineers to determine the representative wall or roof assembly that most closely matches a specific wall or roof assembly. The CTF coefficients and grouping procedure were adopted in the *ASHRAE Handbook—Fundamental* (1989, 1993, and 1997) [9,14,15] and *the ASHRAE Cooling and Heating Load Calculation Manual* [8].

Spitler and Fisher [7,11] developed the across periodic response factors of the representative wall and roof types using the CTF coefficients tabulated in the *ASHRAE Handbook—Fundamental* (1989, 1993 and 1997). However, they found

Table 2
CTF coefficients of a common brick wall

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $b_k$ | 0.386179E–5 | 0.346361E–2 | 0.299762E–1 | 0.341885E–1 | 0.720486E–2 | 0.265935E–3 |
| $d_k$ | 1.000000 | −0.173541$E$1 | 0.931626 | −0.166760 | 0.773671E–2 | 0.000000 |

Table 3
Details of a brick/cavity wall

| Description | Thickness and thermal properties | | | | |
|---|---|---|---|---|---|
| | $L$ [mm] | $\lambda$ [W·m$^{-1}$·K$^{-1}$] | $\rho$ [kg·m$^{-3}$] | $C_P$ [J·kg$^{-1}$·K$^{-1}$] | $R$ [m$^2$·K·W$^{-1}$] |
| Outside surface film | | | | | 0.060 |
| Brickwork | 105 | 0.840 | 1700 | 800 | 0.125 |
| Cavity | | | | | 0.180 |
| Heavyweight concrete | 100 | 1.630 | 2300 | 1000 | 0.06135 |
| Inside surface film | | | | | 0.120 |

Table 4
CTF coefficients of a brick/cavity wall

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $a_k$ | 9.548397 | −18.528113 | 10.569717 | −1.575632 | 0.062597 | −0.000339 |
| $b_k$ | 0.000179 | 0.013915 | 0.043460 | 0.018036 | 0.001034 | 0.000005 |
| $c_k$ | 6.953625 | −12.223156 | 5.985915 | −0.660046 | 0.020334 | −0.000044 |
| $d_k$ | 1.000000 | −1.620834 | 0.726131 | −0.065025 | 0.001594 | 0.000000 |

Table 5
Hourly cross heat flow $\vartheta_Y(j)$ (W·m$^{-2}$) of the common brick wall

| $j$ [h] | $\vartheta_Y(j)$ [W·m$^{-2}$] | $j$ [h] | $\vartheta_Y(j)$ [W·m$^{-2}$] | $j$ [h] | $\vartheta_Y(j)$ [W·m$^{-2}$] |
|---|---|---|---|---|---|
| 0 | 0.00001 | 24 | 0.02104 | 48 | 0.00107 |
| 1 | 0.00350 | 25 | 0.01859 | 49 | 0.00095 |
| 2 | 0.03597 | 26 | 0.01642 | 50 | 0.00084 |
| 3 | 0.09343 | 26 | 0.01450 | 51 | 0.00074 |
| 4 | 0.13637 | 28 | 0.01281 | 52 | 0.00065 |
| 5 | 0.15587 | 28 | 0.01131 | 53 | 0.00058 |
| 6 | 0.15876 | 30 | 0.00100 | 54 | 0.00051 |
| 7 | 0.15231 | 31 | 0.00883 | 55 | 0.00045 |
| 8 | 0.14137 | 32 | 0.00780 | 56 | 0.00040 |
| 9 | 0.12871 | 33 | 0.00689 | 57 | 0.00035 |
| 10 | 0.11584 | 34 | 0.00609 | 58 | 0.00031 |
| 11 | 0.10353 | 35 | 0.00538 | 59 | 0.00027 |
| 12 | 0.09212 | 36 | 0.00475 | 60 | 0.00024 |
| 13 | 0.08175 | 37 | 0.00419 | 61 | 0.00021 |
| 14 | 0.07242 | 38 | 0.00370 | 62 | 0.00019 |
| 15 | 0.06409 | 39 | 0.00327 | 63 | 0.00017 |
| 16 | 0.05667 | 40 | 0.00289 | 64 | 0.00015 |
| 17 | 0.05010 | 41 | 0.00255 | 65 | 0.00013 |
| 18 | 0.04427 | 42 | 0.00226 | 66 | 0.00011 |
| 19 | 0.03912 | 43 | 0.00199 | 67 | 0.00010 |
| 20 | 0.03456 | 44 | 0.00176 | 68 | 0.00010 |
| 21 | 0.03053 | 45 | 0.00155 | 69 | 0.00008 |
| 22 | 0.02697 | 46 | 0.00137 | 70 | 0.00007 |
| 23 | 0.02382 | 47 | 0.00121 | 71 | 0.00006 |

Table 6
Comparisons between the periodic response factors

| Wall | Common brick wall | | Brick/cavity wall | | | | | |
|---|---|---|---|---|---|---|---|---|
| $j$ | $Y_P(j)^a$ | $Y_P(j)^b$ | $X_P(j)^a$ | $X_P(j)^b$ | $Y_P(j)^a$ | $Y_P(j)^b$ | $Z_P(j)^a$ | $Z_P(j)^b$ |
| 0 | 0.02216 | 0.02204 | 9.537769 | 9.537784 | 0.016133 | 0.016134 | 6.929435 | 6.929438 |
| 1 | 0.02309 | 0.02293 | −3.060947 | −3.060951 | 0.028067 | 0.028068 | −0.973495 | −0.973493 |
| 2 | 0.05327 | 0.05318 | −1.318038 | −1.318031 | 0.078398 | 0.078398 | −0.625399 | −0.625395 |
| 3 | 0.10871 | 0.10859 | −0.869076 | −0.869076 | 0.125744 | 0.125746 | −0.516197 | −0.516193 |
| 4 | 0.14987 | 0.14977 | −0.603208 | −0.603209 | 0.149718 | 0.149719 | −0.436559 | −0.436557 |
| 5 | 0.16780 | 0.16770 | −0.427794 | −0.427803 | 0.156418 | 0.156420 | −0.371919 | −0.371921 |
| 6 | 0.16929 | 0.16920 | −0.309788 | −0.309800 | 0.152865 | 0.152866 | −0.318390 | −0.318394 |
| 7 | 0.16162 | 0.16153 | −0.229322 | −0.229332 | 0.143724 | 0.143724 | −0.273560 | −0.273564 |
| 8 | 0.14958 | 0.14977 | −0.173603 | −0.173610 | 0.131885 | 0.131886 | −0.235691 | −0.235695 |
| 9 | 0.13597 | 0.13587 | −0.134327 | −0.134331 | 0.119092 | 0.119093 | −0.203486 | −0.203490 |
| 10 | 0.12225 | 0.12215 | −0.106081 | −0.106083 | 0.106364 | 0.106365 | −0.175956 | −0.175958 |
| 11 | 0.10919 | 0.10907 | −0.085324 | −0.085324 | 0.094269 | 0.094270 | −0.152327 | −0.152329 |
| 12 | 0.09712 | 0.09699 | −0.069725 | −0.069725 | 0.083094 | 0.083095 | −0.131986 | −0.131988 |
| 13 | 0.08617 | 0.08602 | −0.057740 | −0.057739 | 0.072957 | 0.072957 | −0.114435 | −0.114437 |
| 14 | 0.07632 | 0.07617 | −0.048336 | −0.048336 | 0.063874 | 0.063874 | −0.099266 | −0.099267 |
| 15 | 0.06753 | 0.06738 | −0.040816 | −0.040816 | 0.055806 | 0.055806 | −0.086139 | −0.086139 |
| 16 | 0.05972 | 0.05956 | −0.034700 | −0.034700 | 0.048683 | 0.048683 | −0.074767 | −0.074767 |
| 17 | 0.05279 | 0.05262 | −0.029657 | −0.029657 | 0.042421 | 0.042421 | −0.064909 | −0.064909 |
| 18 | 0.04665 | 0.04649 | −0.025448 | −0.025449 | 0.036935 | 0.036935 | −0.056360 | −0.056360 |
| 19 | 0.04122 | 0.04106 | −0.021904 | −0.021905 | 0.032139 | 0.032138 | −0.048941 | −0.048941 |
| 20 | 0.03641 | 0.03626 | −0.018897 | −0.018899 | 0.027953 | 0.027952 | −0.042502 | −0.042503 |
| 21 | 0.03217 | 0.03202 | −0.016331 | −0.016333 | 0.024304 | 0.024303 | −0.036913 | −0.036913 |
| 22 | 0.02842 | 0.02827 | −0.014131 | −0.014134 | 0.021126 | 0.021127 | −0.032060 | −0.032060 |
| 23 | 0.02510 | 0.02496 | −0.012240 | −0.012243 | 0.018361 | 0.018360 | −0.027846 | −0.027847 |
| $\sum$ | 2.02243 | 2.01934 | 1.830336 | 1.830298 | 1.830330 | 1.830340 | 1.830332 | 1.830318 |
| $U$ | 2.02243 | | 1.830330 | | | | | |

[a] Based on the FDR method.
[b] Based on CTF coefficients.

Table 7
Periodic response factors of problematic ASHRAE representative wall or roof types

| $j$ | $Y_P(j)^{a}$ | $Y_P(j)^{b}$ | $Y_P(j)^{a}$ | $Y_P(j)^{b}$ | $Y_P(j)^{a}$ | $Y_P(j)^{b}$ | $Y_P(j)^{a}$ | $Y_P(j)^{b}$ |
|---|---|---|---|---|---|---|---|---|
| | Wall 30 | | Wall 31 | | Wall 35 | | Wall 36 | |
| 0 | 0.008675 | 0.008359 | 0.005491 | 0.005447 | 0.011023 | 0.008953 | 0.015442 | 0.015236 |
| 1 | 0.007813 | 0.007494 | 0.005183 | 0.004959 | 0.010549 | 0.008625 | 0.015056 | 0.014875 |
| 2 | 0.007054 | 0.006735 | 0.004657 | 0.004517 | 0.010259 | 0.008307 | 0.014727 | 0.014543 |
| 3 | 0.006742 | 0.006422 | 0.004331 | 0.004244 | 0.010030 | 0.008057 | 0.014626 | 0.014439 |
| 4 | 0.007639 | 0.007316 | 0.004593 | 0.004513 | 0.010052 | 0.008066 | 0.015052 | 0.014863 |
| 5 | 0.009959 | 0.009632 | 0.005645 | 0.005547 | 0.010478 | 0.008487 | 0.015927 | 0.015736 |
| 6 | 0.013084 | 0.012755 | 0.007266 | 0.007118 | 0.011228 | 0.009241 | 0.016927 | 0.016737 |
| 7 | 0.016218 | 0.015883 | 0.009017 | 0.008798 | 0.012088 | 0.010108 | 0.017812 | 0.017622 |
| 8 | 0.018822 | 0.018478 | 0.010527 | 0.010256 | 0.012878 | 0.010902 | 0.018487 | 0.018294 |
| 9 | 0.020663 | 0.020311 | 0.011631 | 0.011330 | 0.013504 | 0.011525 | 0.018935 | 0.018741 |
| 10 | 0.021720 | 0.021364 | 0.012335 | 0.011989 | 0.013933 | 0.011948 | 0.019180 | 0.018990 |
| 11 | 0.022084 | 0.021729 | 0.012712 | 0.012271 | 0.014164 | 0.012184 | 0.019261 | 0.019076 |
| 12 | 0.021888 | 0.021538 | 0.012828 | 0.012245 | 0.014219 | 0.012262 | 0.019213 | 0.019036 |
| 13 | 0.021269 | 0.020926 | 0.012719 | 0.011985 | 0.014135 | 0.012217 | 0.019069 | 0.018898 |
| 14 | 0.020349 | 0.020014 | 0.012407 | 0.011555 | 0.013951 | 0.012077 | 0.018854 | 0.018688 |
| 15 | 0.019230 | 0.018903 | 0.011922 | 0.011099 | 0.013703 | 0.011867 | 0.018587 | 0.018423 |
| 16 | 0.017994 | 0.017673 | 0.011305 | 0.010393 | 0.013415 | 0.011607 | 0.018284 | 0.018120 |
| 17 | 0.016702 | 0.016387 | 0.010602 | 0.009737 | 0.013105 | 0.011312 | 0.017954 | 0.017790 |
| 18 | 0.015402 | 0.015089 | 0.009857 | 0.009067 | 0.012786 | 0.010944 | 0.017607 | 0.017441 |
| 19 | 0.014125 | 0.013815 | 0.009103 | 0.008400 | 0.012462 | 0.010660 | 0.017248 | 0.017080 |
| 20 | 0.012897 | 0.012586 | 0.008364 | 0.007750 | 0.012137 | 0.010319 | 0.016882 | 0.016712 |
| 21 | 0.011731 | 0.011418 | 0.007655 | 0.007125 | 0.011813 | 0.009974 | 0.016513 | 0.016341 |
| 22 | 0.010636 | 0.010322 | 0.006988 | 0.006531 | 0.011489 | 0.009630 | 0.016143 | 0.015970 |
| 23 | 0.009618 | 0.009301 | 0.006369 | 0.005971 | 0.011167 | 0.009289 | 0.015774 | 0.015601 |
| $\sum$ | 0.352315 | 0.34445 | 0.213505 | 0.202847 | 0.294569 | 0.248561 | 0.413557 | 0.409252 |
| $U$ | 0.352270 | | 0.213476 | | 0.294528 | | 0.413499 | |
| $e^{c}$ | 0.13% | 2.22% | 0.136% | 4.98% | 0.14% | 15.61% | 0.14% | 1.03% |
| | Wall 37 | | Wall 38 | | Roof 37 | | Roof 38 | |
| 0 | 0.008025 | 0.005075 | 0.008630 | 0.005342 | 0.007799 | 0.007526 | 0.007374 | 0.006529 |
| 1 | 0.007217 | 0.004578 | 0.007919 | 0.004934 | 0.007433 | 0.007211 | 0.006873 | 0.006304 |
| 2 | 0.006856 | 0.004101 | 0.007615 | 0.004541 | 0.007146 | 0.006915 | 0.006675 | 0.006088 |
| 3 | 0.006542 | 0.003678 | 0.007349 | 0.004196 | 0.006982 | 0.006744 | 0.006526 | 0.005925 |
| 4 | 0.006414 | 0.003464 | 0.007257 | 0.004046 | 0.007196 | 0.006954 | 0.006584 | 0.005973 |
| 5 | 0.006675 | 0.003668 | 0.007527 | 0.004285 | 0.007867 | 0.007624 | 0.006962 | 0.006347 |
| 6 | 0.007367 | 0.004333 | 0.008189 | 0.004939 | 0.008791 | 0.008549 | 0.007572 | 0.006960 |
| 7 | 0.008337 | 0.005295 | 0.009082 | 0.005839 | 0.009712 | 0.009473 | 0.008230 | 0.007621 |
| 8 | 0.009374 | 0.006341 | 0.010006 | 0.006779 | 0.010475 | 0.010235 | 0.008800 | 0.008192 |
| 9 | 0.010329 | 0.007307 | 0.010822 | 0.007608 | 0.011020 | 0.010777 | 0.009225 | 0.008612 |
| 10 | 0.011118 | 0.008096 | 0.011464 | 0.008251 | 0.011343 | 0.011100 | 0.009491 | 0.008874 |
| 11 | 0.011693 | 0.008671 | 0.011898 | 0.008687 | 0.011471 | 0.011233 | 0.009609 | 0.008998 |
| 12 | 0.012028 | 0.009032 | 0.012112 | 0.008929 | 0.011443 | 0.011216 | 0.009605 | 0.009010 |
| 13 | 0.012123 | 0.009198 | 0.012119 | 0.009003 | 0.011297 | 0.011084 | 0.009510 | 0.008939 |
| 14 | 0.012011 | 0.009196 | 0.011957 | 0.008941 | 0.011070 | 0.010869 | 0.009353 | 0.008806 |
| 15 | 0.011740 | 0.009056 | 0.011675 | 0.008771 | 0.010787 | 0.010595 | 0.009158 | 0.008630 |
| 16 | 0.011357 | 0.008805 | 0.011318 | 0.008519 | 0.010470 | 0.010283 | 0.008941 | 0.008425 |
| 17 | 0.010903 | 0.008467 | 0.010920 | 0.008205 | 0.010132 | 0.009946 | 0.008712 | 0.008201 |
| 18 | 0.010413 | 0.008063 | 0.010509 | 0.007845 | 0.009784 | 0.009595 | 0.008478 | 0.007965 |
| 19 | 0.009913 | 0.007610 | 0.010099 | 0.007453 | 0.009432 | 0.009239 | 0.008242 | 0.007723 |
| 20 | 0.009418 | 0.007124 | 0.009702 | 0.007040 | 0.009080 | 0.008884 | 0.008005 | 0.007479 |
| 21 | 0.008938 | 0.006617 | 0.009319 | 0.006616 | 0.008733 | 0.008532 | 0.007770 | 0.007236 |
| 22 | 0.008475 | 0.006101 | 0.008949 | 0.006187 | 0.008391 | 0.008187 | 0.007536 | 0.006996 |
| 23 | 0.008030 | 0.005585 | 0.008590 | 0.005761 | 0.008058 | 0.007852 | 0.007305 | 0.006760 |
| $\sum$ | 0.225296 | 0.159461 | 0.235029 | 0.162717 | 0.225911 | 0.220623 | 0.196537 | 0.182593 |
| $U$ | 0.225264 | | 0.234996 | | 0.225883 | | 0.196509 | |
| $e^{c}$ | 0.142% | 29.21% | 0.14% | 30.76% | 0.124% | 2.33% | 0.142% | 7.08% |

[a] Based on the FDR method.
[b] Based on CTF coefficients.
[c] $e = |(\sum - U)/U| \times 100\%$.

that the conduction transfer function coefficients originally published in the *ASHRAE Handbook* were inaccurate for a few of the very high mass walls and roofs [11]. The errors can be qualified by checking whether or not the CTF coefficients satisfy a fundamental relationship between the $U$-factor and the CTF coefficients:

$$U = \frac{\sum_{i=0} a_i}{1 + \sum_{i=1} d_i} = \frac{\sum_{i=0} b_i}{1 + \sum_{i=1} d_i} = \frac{\sum_{i=0} c_i}{1 + \sum_{i=1} d_i} \tag{28}$$

The wall surface for which the discrepancy between the actual $U$-factor and the $U$-factor calculated using CTF coefficients exceeding 1% are Roof 37 (2.33%), Roof 38 (7.08%), Wall 30 (2.22%), Wall 31 (4.98%), Wall 35 (15.61%), Wall 36 (1.03%), Wall 37 (29.21%), and Wall 38 (30.76%), which are very close to the errors presented by Spitler [11]. In all the cases, the $U$-factor based on the CTF coefficients is lower than the actual $U$-factor. The periodic response factors of these eight wall or roof types, determined using the new procedure, are given in Table 7 for comparison. Their periodic response factors are also calculated using their CTF coefficients, which are listed in the same table. The largest discrepancy between the $U$-factor and the sum of the periodic response factors calculated by the new procedure is 0.142%, much less than that using the CTF coefficients. Obviously, the accuracy of the new procedure is much greater. The discrepancy between the $U$-factor and the sum of the periodic response factors using the new procedure was mainly caused by unit conversion, as the coefficients listed in the ASHRAE handbooks were converted from the original data in Imperial units.

## 4. Application

When design engineers utilize the RTSM to conduct design load calculations, they can apply the procedure based on the FDR method in two ways to generate the periodic response factors of practical walls. One is to apply directly the procedure based on the FDR method, as described in this paper. Another is to use the grouping procedure originally proposed by Harris and McQuiston [12] and later described in the *ASHRAE Fundamentals* [9,14,15] and the *Cooling and Heating Load Calculation Manual* [8]. When using the grouping procedure, the periodic response factors of the typical wall and roof types are calculated and tabulated beforehand using a procedure based on the FDR method. When conducting design load calculations, design engineers apply the grouping procedure to select the typical walls or roofs. However, when using this procedure, a slightly different “unnormalization” procedure is necessary as the practical walls and roofs are normally not exactly the same as those listed in the handbooks. To “unnormalize” the periodic response factors, each $Y_P$ coefficient is multiplied by the ratio of the actual $U$-factor to the $U$-factor of the typical wall or roof. Some error may be caused by using this “unnormalization” procedure.

If RTSM users desire a higher degree of accuracy, the present procedure based on the FDR method provides another advantage, as the periodic response factors for the practical walls and roofs can be generated directly from the thermal and geometric parameters of the walls and roofs by using the new procedure. It is especially applicable to calculate the periodic response factors of new building constructions. This procedure is not only easy to implement, and accurate in calculation, but also, if needed, can be used to generate outside and inside periodic response factor series.

## 5. Conclusions

Hourly design cooling loads are usually calculated using steady periodic inputs, and periodic response factors for conduction heat transfer and thermal zone response can be advantageously utilized in the computational procedure. Currently, there are two procedures used to determine periodic response factors for building elements. One is based on Laplace transforms and direct root-finding procedure, and another is based on CTF coefficients. However, in some cases, these procedures might result in inaccuracy and unreliable periodic response factors. The grouping and “unnormalization” procedure to determine the periodic response factors of the actual walls or roofs based on the representative wall and roof types might not meet the requirements of users in terms of accuracy.

The new calculation procedure based on the FDR method can be implemented easily to generate directly the periodic response factors of multilayer walls and roofs from their geometric and thermal properties. Using this new procedure, a polynomial $s$-transfer function is constructed from the theoretical frequency characteristics of a multilayer wall or roof by solving a set of linear equations. The periodic response factors are generated by calculating the poles and residues of the polynomial $s$-transfer function. The comparisons and validations show that this procedure is an accurate and hopefully better alternative approach. Although the actual periodic response factors can be determined by the grouping and “unnormalization” procedure from the wall types and roof types tabulated in ASHRAE fundamentals, RTSM users can conveniently calculate the outside, across and inside periodic response factors directly on the basis of the geometric and thermal properties, using the procedure based on the FDR method. This gives more accurate periodic response factors. Certainly, the new procedure offers convenience in calculating the periodic response factors of new building constructions or building elements that are composed of new materials.

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